Math Challengers

2026 Regionals Contest

Blitz Stage

This stage of the contest has 26 problems over 4 pages and is to be completed in 40 minutes.
Each correct answer is worth 1 mark for a maximum of 26 marks.

Do not open the contest paper until instructed to do so.

Please fill in your school, grade, and name below.

School: ______________________________

Grade (Please circle): **8** **9** **10**

Team Number (Please circle): **1** **2** **3** **Individual**

Name: ______________________________

The table below is for marker use only.

Marker	Q1-7	Q8-14	Q15-20	Q21-26	Total	Initials
Marker 1						
Marker 2						
Marker 3						

1. Given that $1 + 2 + 3 + 4 + k = 1 \times 2 \times 3 \times 4 \times k$, express the value of k as a common fraction in lowest terms.

1. __________

2. Given that Alice has 4 different pairs of jeans, 3 different dresses, and 5 different shirts, and she either wears a shirt with jeans or a shirt with a dress, in how many ways can Alice dress herself?

2. __________

3. Suppose Bob flies from Vancouver to Seoul to visit some relatives. His flight from Vancouver departs at 12:55am local time, arrives at 4:30am local time, and lasts less than 24 hours. Given that the time in Seoul is 16 hours ahead of the time in Vancouver, determine the length of the flight in minutes.

3. ______min

4. In a herd of sheep with white, brown, gray, and black fleece, 85% of the sheep are not white, 79% of the sheep are not brown, 67% of the sheep are not gray, and k% of the sheep are not black. Determine the value of k.

4. __________

5. By substituting only addition and multiplication operators in place of the circles, determine the maximum possible value of the expression $1 \circ 2 \circ 3 \circ 4$.

5. __________

6. Bob bought some watermelons. He ate half of all of them right away and he ate an additional half a watermelon later in the day. He then had 1 watermelon left. Determine the number of watermelons Bob bought.

6. __________

7. Alice, Bob, Charlie, and David wrote the same math test out of 100. Given that the average of Alice and Bob's scores is 73, the average of Bob and Charlie's scores is 78, and the average of Charlie and David's scores is 85, determine the average of Alice and David's scores.

7. __________

8. David wrote 4 out of the 7 tests in his physics class and achieved an average of 91% across them. Given that he wishes to have an average of 93% across all 7 tests, determine the minimum percentage he must achieve on the 5th test for this to be possible.

8. ________%

9. The points $(7, -2)$ and $(-4, 5)$ are the endpoints of a diagonal of a square. Determine the area of this square.

9. __________

10. In right triangle ABC, we have $\angle B = 90^\circ$ and $AB = 6, BC = 10$. Suppose point D is drawn such that $\angle DAC = 90^\circ$ and $AD = 15$. Determine the length of CD.

10. __________

11. Alice and Bob each roll 2 fair die and sum up their rolls. Determine the probability that they obtain the same sum. Express your answer as a common fraction in lowest terms.

11. __________

12. Assume a, b, c are real numbers such that $ab = 20$, $ac = 30$, and $bc = 60$. Determine the value of $a^2 + b^2 + c^2$.

12. __________

13. Suppose x is a real number such that $x^x = 3$. Determine the value of $x^{2x^{x+1}}$.

13. __________

14. Suppose that for all nonnegative integers n, circle C_n is inscribed in square S_n and square S_n is inscribed in circle C_{n+1}. Given that the side length of S_0 is 2048, determine the positive integer k such that S_k has an area of 2^{2026}.

14. __________

15. A palindrome is a number that reads the same forwards and backwards. Determine the product of the digits of smallest 7-digit palindrome that does not contain the digits 0 or 1 and is divisible by 36.

15. __________

16. Given that a randomly chosen 3-digit integer has at least 2 odd digits, determine the probability that it has exactly 2 odd digits. Express your answer as a common fraction in lowest terms.

16. __________

17. Charlie has 37 fence panels, each of which is 2 meters long. He wishes to enclose a rectangular area on 3 sides since the wall of a cliff encloses the last side. Given that Charlie does not cut his panels, determine the maximum area that can be enclosed, in square meters.

17. ________m^2

18. Suppose a and b are real numbers such that $a + b = 4$ and $a^2 + b^2 = 9$. Determine the value of $a^5 + b^5$.

18. __________

19. The intersections between the lines $y = m_1x+b_1$, $y = m_1x+b_2$, $y = m_2x+b_1$, and $y = m_2x+b_2$ are the vertices of a parallelogram. Given that $b_2 - b_1 = 12$ and $m_2 - m_1 = 8$, determine the area of the parallelogram.

19. __________

20. A lightbulb is suspended from the center of the ceiling of a square room. Given that the lightbulb is 7 meters, 9 meters, and 6 meters from the center of the floor, a corner at the floor, and a corner at the ceiling, respectively, determine the volume of the room in cubic meters.

20. ________m^3

21. Evaluate the expression

$$(\sqrt{13}+\sqrt{14}+\sqrt{15})(\sqrt{13}+\sqrt{14}-\sqrt{15})(\sqrt{13}-\sqrt{14}+\sqrt{15})(-\sqrt{13}+\sqrt{14}+\sqrt{15})$$

21. __________

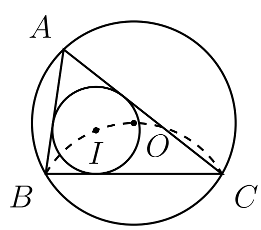
22. There are 2 blue marbles, 3 red marbles, and 5 yellow marbles in a bag. Alice draws marbles from this bag at random. Determine the probability that Alice requires at least 8 draws to get all 3 colors of marbles. Express your answer as a common fraction.

22. __________

23. The quadratic $q_1(x)$ has leading coefficient 1 and its vertex $(2,-7)$, and the quadratic $q_2(x)$ has leading coefficient -1 and its vertex at $(-4,k)$. Given that the graphs of $y=q_1(x)$ and $y=q_2(x)$ are tangent, determine the value of k.

23. __________

24. Triangle ABC has circumcentre O and incentre I. Given that quadrilateral $BIOC$ is cyclic, determine the measure of $\angle A$ in degrees.



24. __________°

25. Evaluate the sum $100^3-99^3+98^3-97^3+\cdots+2^3-1^3$.

25. __________

26. For some positive integer $n \le 2026$, Alice chooses a number a_k from the set $\left\{\dfrac{k}{k+1}, \dfrac{k+1}{k}\right\}$ for all positive integers $k \le n$. Determine the number of possible values of n such that Alice may choose her numbers $a_1, a_2, \ldots, a_n$ such that the product $a_1 a_2 \ldots a_n = 1$.

26. __________

Math Challengers

2026 Regionals Contest

Bull's Eye Stage

This stage of the contest has a total of 12 problems over 3 pages. Each correct answer is worth 2 marks for a maximum of 24 marks. This stage is divided into 3 sections with 4 questions each and each section is to be done in 12 minutes.

Working ahead or looking at previous sections when the time is up is prohibited. Do not open the contest paper until instructed to do so.

Please fill in your school, grade, and name below.

School: ______________________________

Grade (Please circle): **8** **9** **10**

Team Number (Please circle): **1** **2** **3** **Individual**

Name: ______________________________

The table below is for marker use only.

Marker	Q1-4	Q5-8	Q9-12	Total	Initials
Marker 1					
Marker 2					
Marker 3					

1. Given that 2 cows and 3 goats have the same value as 1 cow and 5 goats, then 2026 goats is equivalent to k cows. Determine the value of k.

1. __________

2. A band teacher realizes that he can arrange his class into x rows with y students in each row as well as $x - 3$ rows with $y + 4$ students in each row. Given there are between 50 and 100 students in the marching band, determine number of students in the marching band.

2. __________

3. Suppose $\{a_n\}$ and $\{b_n\}$ are arithmetic sequences with common differences d_a and d_b, respectively such that $a_{20} = b_{26}$ and $a_{202} = b_6$. Determine the ratio $\frac{d_b}{d_a}$. Express your answer as a common fraction in lowest terms.

3. __________

4. A rectangular prism has sides of integer lengths and a volume of 126. A new rectangular prism is created by extending all the sides of the previous rectangular prism by 1. Determine the minimum possible volume of the new prism.

4. __________

5. In a group of 10 students, a teacher selects 1 to be president, 2 to be vice presidents, and 3 to be secretaries. Given that no student has more than 1 role, determine the number of ways this can this be done.

5. __________

6. Alice has 6 square tiles with identical dimensions, 2 black and 4 white. 2 arrangements are considered the same if one can be rotated to obtain the other. Determine the number of different arrangements that Alice can make.

6. __________

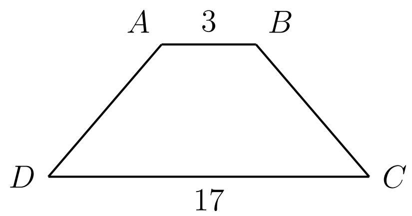
7. Determine the number of positive integers n such that $n^3 + 7n^2 - 5n + 13$ is a multiple of $n - 5$.

7. __________

8. Assume n is a positive integer with 18 distinct positive factors, while n^2 has 75 distinct positive factors. Determine the number of distinct positive factors of n^3.

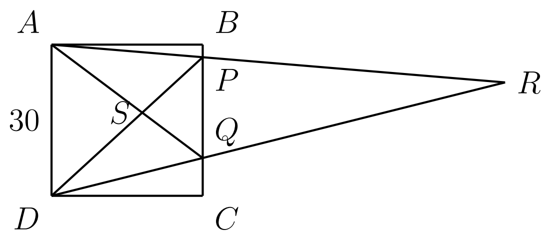
8. __________

9. An isosceles trapezoid has $AB \parallel CD$, $AB = 3$, $CD = 17$, and an area of 240. Determine the perimeter of $ABCD$.



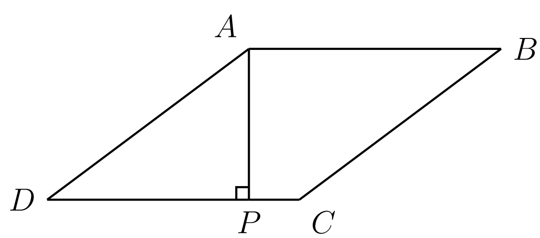
9. ________

10. Square $ABCD$ has side length 30. There exists points P and Q on BC such that $PQ = 20$. Let R be the intersection of AP and DQ and let S be the intersection of AQ and DP. Determine the area of quadrilateral $PRQS$.



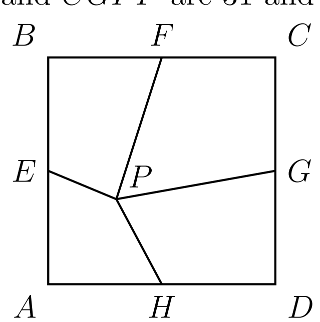
10. ________

11. In rhombus $ABCD$, suppose that point P is on CD such that $AP \perp CD$. Given that $AB = 5PC$, then determine the value of $\dfrac{BD}{AC}$.



11. ________

12. In square $ABCD$, assume points E, F, G, H are the midpoints of AB, BC, CD, AD, respectively. Let P be a point in the interior of $ABCD$. Given that the areas of quadrilaterals $AEPH$ and $CGPF$ are 31 and 67, respectively, determine the length of AB.



12. ________

Math Challengers

2026 Regionals Contest

Co-op Stage

This stage of the contest has 15 problems over 3 pages and is to be completed in 36 minutes. Each correct answer is worth 2 marks for a maximum of 30 marks. Each team will receive 1 white copy per team member and 1 yellow copy of this stage.

Please note that only answers on the yellow sheet will be marked.

Do not open the contest paper until instructed to do so.

Please fill in your school, grade, and names of all your team members below.

School: ______________________________

Grade (Please circle): **8** **9** **10**

Team Number (Please circle): **1** **2** **3**

Name of each team member (up to 5): ______________________________

____

The table below is for marker use only.

Marker	Q1-5	Q6-10	Q11-15	Total	Initials
Marker 1					
Marker 2					
Marker 3					

1. Evaluate $2 + \dfrac{1}{0 + \frac{1}{2+\frac{1}{6}}}$. Express your answer as a common fraction in lowest terms.

1. ___________

2. If x and y are real numbers such that $\dfrac{3x+4}{5y+6} = \dfrac{2}{3}$, determine the value of $\dfrac{y}{x}$. Express your answer as a common fraction in lowest terms.

2. ___________

3. The ages of Mary and Jane are in the ratio 5:7. In 4 years, their ages will be in the ratio 3:4. Determine Jane's current age.

3. ___________

4. Alice and Bob each ran 100 times around the same track at constant speeds. Alice took 2 minutes to complete each lap, while Bob took 2 minutes 15 seconds. Over the course of the 100 laps, determine the number of times that Alice passed Bob.

4. ___________

5. A bag contains 5 red and 7 yellow marbles. Alice draws 3 marbles from the bag without replacement. Determine the probability that the third marble is red. Express your answer as a fraction in lowest terms.

5. ___________

6. Determine the area enclosed by the graph of the equation $(2x - y - 5)(x + y - 7)(x + 2) = 0$.

6. __________

7. Suppose $\{a_n\}$ is a sequence satisfying $a_1 = 10$ and $a_{n+1} = \dfrac{a_n + 1}{1 - a_n}$. Determine a_{2026}. Express your answer as a common fraction in lowest terms.

7. __________

8. Alice and Bob were walking to their boarding gates at an airport with constant speeds along parallel, straight paths. At $t = 0$ seconds, Alice passes Bob. At $t = 40$ seconds, Bob gets on a long conveyor belt that travels at 2m/s while continuing to walk at his constant speed relative to the conveyor. At $t = 90$ seconds, Bob passes Alice. Determine the difference between Alice and Bob's speeds in km/h.

8. _____km/h

9. Given that $\dfrac{17}{20} < \dfrac{p}{q} < \dfrac{6}{7}$, where p and q are positive integers, determine the minimum possible value of $p + q$.

9. __________

10. The equation $3^{4052x+2} + 81 = 3^{2026x+5}$ has real solutions x_1 and x_2. Determine the value of $\dfrac{1}{x_1 + x_2}$.

10. __________

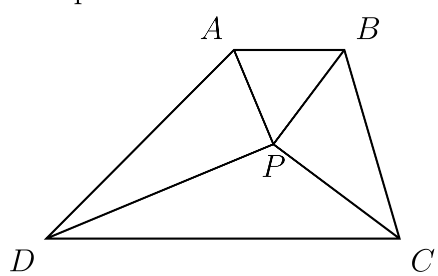
11. Determine the remainder when $9^{2026} + 11^{2026}$ is divided by 1000.

11. __________

12. Suppose the line $y = mx - b$ is the image of the line $y = 7x + 9$ after reflecting about the line $y = 12x - 11$. Determine the value of $m + b$.

12. __________

13. In trapezoid $ABCD$, $AB||CD$ and the angle bisectors of $\angle A$, $\angle B$, $\angle C$, and $\angle D$ all meet at P. Given that $\dfrac{AP}{DP} = \dfrac{5}{12}$, $\dfrac{BP}{CP} = \dfrac{3}{4}$, and the area of trapezoid $ABCD$ is 90, determine the perimeter of trapezoid $ABCD$.



13. __________

14. Let S be the infinite sum $\dfrac{1}{9} + \dfrac{1}{99} + \dfrac{1}{999} + \cdots$. Determine the remainder when $\lfloor 10^{2026} S \rfloor$ is divided by 1000, where $\lfloor x \rfloor$ is the greatest integer that is at most x.

14. __________

15. Given that positive real x and y satisfies the system

$$\begin{aligned}(\sqrt{x}+4)(\sqrt{y}+4) &= 97\\ (\sqrt{x}+\sqrt{y}-3)(\sqrt{xy}-3) &= 260\end{aligned}$$

determine the value of $x + y$.

15. __________

Answer Key

Blitz

1. $\frac{10}{23}$
2. 35
3. 695
4. 69
5. 25
6. 3
7. 80
8. 87
9. 85
10. 19
11. $\frac{73}{648}$
12. 140
13. 729
14. 2004
15. 576
16. $\frac{14}{19}$
17. 684
18. 149
19. 18
20. 576
21. 584
22. $\frac{3}{40}$
23. 11
24. 60
25. 507500
26. 44

Bull's Eye

1. 1013
2. 72
3. $-\frac{91}{10}$
4. 224
5. 12600
6. 18
7. 22
8. 196
9. 70
10. 720
11. 3
12. 14

Co-op

1. $\frac{25}{6}$
2. $\frac{9}{10}$
3. 28
4. 11
5. $\frac{5}{12}$
6. 54
7. $-\frac{11}{9}$
8. 4
9. 50
10. 1013
11. 2
12. 168
13. 42
14. 754
15. 111

Solutions

Blitz

1. Given that $1+2+3+4+k = 1\times 2\times 3\times 4\times k$, express the value of k as a common fraction in lowest terms.

Solution

Simplifying, we get $k + 10 = 24k$, from which $k = \boxed{\frac{10}{23}}$.

2. Given that Alice has 4 different jeans, 3 different dresses, and 5 different shirts, and she either wears a shirt with jeans or a shirt with a dress, in how many ways can Alice dress herself?

Solution

If Alice wears jeans, then she can dress herself in $5\cdot 4 = 20$ ways. If Alice wears a dress, then she can dress herself in $5 \cdot 3 = 15$ ways. In total, Alice can dress herself in $20 + 15 = \boxed{35}$ ways.

3. Suppose Bob flies from Vancouver to Seoul to visit some relatives. His flight from Vancouver departed at 12:55am local time and arrived at 4:30am local time. Given that the time in Seoul is 16 hours ahead of the time in Vancouver, determine the length of the flight in minutes.

Solution

Since Seoul is ahead by 16 hours, 4:30am in Seoul is 12:30pm in Vancouver. The difference between 12:55am and 12:30pm is 11 hours, 35 minutes or $\boxed{695}$ minutes.

4. In a herd of sheep with white, brown, gray, and black fur, 85% of the sheep are not white, 79% of the sheep are not brown, 67% of the sheep are not gray, and k% of the sheep are not black. Determine the value of k.

Solution

Since 85% of the sheep are not white, 15% of the sheep are white.

Similarly, $100\% - 79\% = 21\%$ of the sheep are brown and $100\% - 67\% = 33\%$ of the sheep are gray. Therefore, $15\% + 21\% + 33\% = 69\%$, so $k = \boxed{69}$.

5. By placing only addition and multiplication operators in place of the circles, determine the maximum possible value of the expression $1 \circ 2 \circ 3 \circ 4$.

Solution
The maximum is $1 + 2 \cdot 3 \cdot 4 = \boxed{25}$.

6. One morning, Bob buys some watermelons while grocery shopping. He ate half his watermelons when he returned home and he ate an additional half a watermelon a few hours later. At the end of that day, Bob had 1 watermelon remaining from his grocery trip that morning. Determine the number of watermelons Bob bought.

Solution
Suppose Bob bought x watermelons. That day, Bob ate $\frac{x}{2} + \frac{1}{2}$ watermelons and was left with 1 watermelon. Therefore, we have $x = \frac{x}{2} + \frac{1}{2} + 1$, from which we get $\frac{x}{2} = \frac{3}{2}$ or $x = \boxed{3}$.

7. Alice, Bob, Charlie, and David write the same math test out of 100. Given that the average of Alice and Bob's scores is 73, the average of Bob and Charlie's scores is 78, and the average of Charlie and David's scores is 85, determine the average of Alice and David's scores.

Solution
Let a, b, c, and d be Alice, Bob, Charlie, and David's scores, respectively. We are given

$$\frac{a+b}{2} = 73 \tag{1}$$
$$\frac{b+c}{2} = 78 \tag{2}$$
$$\frac{c+d}{2} = 85 \tag{3}$$

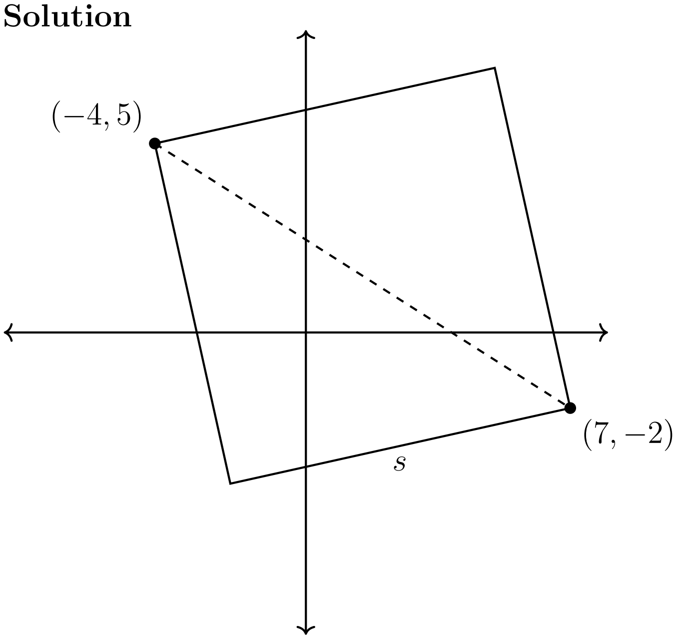
Noting that $\frac{a+d}{2} = (1) + (3) - (2)$, the correct answer is $73 + 85 - 78 = \boxed{80}$.

8. David has written 4 out of the 7 tests in his physics class and has achieved an average of 91% across them. Given that he wishes to have an average of 93% across all 7 tests, determine the minimum percentage he must achieve on the 5th test for this to be possible.

Solution
In order to minimize the score on the 5th test, assume David gets 100% on both his 6th and 7th tests. His 5th test mark is then forced to be $7 \cdot 93 - 4 \cdot 91 - 2 \cdot 100 = \boxed{87\%}$.

9. The points $(7, -2)$ and $(-4, 5)$ are the endpoints of a diagonal of a square. Determine the area of this square.

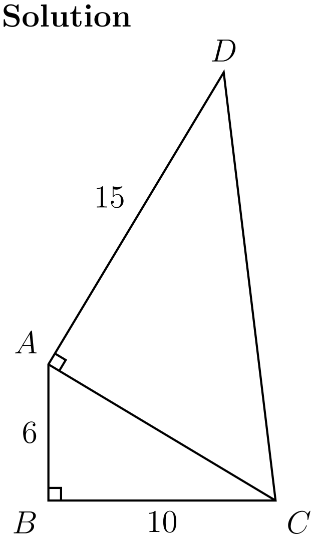
Solution



Suppose the side length of the square is s. By the Pythagorean theorem, we have $(7+4)^2 + (5+2)^2 = 170 = 2s^2$. Therefore, the desired area is $s^2 = \dfrac{170}{2} = \boxed{85}$.

10. In right triangle ABC, we have $\angle B = 90^\circ$ and $AB = 6, BC = 10$. Suppose point D is drawn such that $AD \perp AC$ and $AD = 15$. Determine the length of CD.

Solution



By the Pythagorean theorem, we have $CD^2 = AC^2 + AD^2 = AB^2 + BC^2 + AD^2$. Substituting the given lengths, $CD^2 = 6^2 + 10^2 + 15^2 = 361$, so $CD = \sqrt{361} = \boxed{19}$.

11. Alice and Bob each roll 2 fair die and sum up their rolls. Determine the probability that they get the same sum. Express your answer as a common fraction in lowest terms.

Solution
For integers $2 \le n \le 12$, let p_n be the probability that the sum of the dice is n. The desired probability is then $\sum_{k=2}^{12} p_k^2 = \dfrac{1^2+2^2+\cdots+6^2+5^2+\cdots+1^2}{36^2} = \boxed{\dfrac{73}{648}}$.

12. Assume a, b, c are real numbers such that $ab = 20$, $ac = 30$, and $bc = 60$. Determine the value of $a^2 + b^2 + c^2$.

Solution
Multiplying the 3 equations together, we have $a^2b^2c^2 = 20 \cdot 30 \cdot 60$. Therefore, we have $c^2 = \dfrac{a^2b^2c^2}{a^2b^2} = \dfrac{20 \cdot 30 \cdot 60}{20^2} = 90$. Similarly, $b^2 = \dfrac{20 \cdot 30 \cdot 60}{30^2} = 40$, and $a^2 = \dfrac{20 \cdot 30 \cdot 60}{60^2} = 10$, so $a^2 + b^2 + c^2 = 90 + 40 + 10 = \boxed{140}$.

13. Suppose x is a real number such that $x^x = 3$. Determine the value of $x^{2x^{x+1}}$.

Solution
We have $x^{2x^{x+1}} = x^{2x^x \cdot x} = x^{6x} = (x^x)^6 = 3^6 = \boxed{729}$.

14. Suppose that for all nonnegative integers n, circle C_n is inscribed in square S_n and square S_n is inscribed in circle C_{n+1}. Given that the side length of S_0 is 2048, determine the positive integer k such that S_k has an area of 2^{2026}.

Solution
For some nonnegative integer a, consider S_a, C_{a+1}, and S_{a+1}. Note that the diameter of C_{a+1} is equal to both the diagonal of S_a and the side length of S_{a+1}. We therefore have that the side length of S_{a+1} is $\sqrt{2}$ times longer than that of S_a, so S_{a+1} has double the area as S_a.
Let $A(S_n)$ be the area of S_n. Since we have shown that $A(S_{n+1}) = 2A(S_n)$, it follows that $A(S_n) = 2^n A(S_0)$ or $\dfrac{A(S_n)}{A(S_0)} = 2^n$. Since we are given that $A(S_0) = (2^{11})^2 = 2^{22}$ and $A(S_k) = 2^{2026}$, we have $2^k = \dfrac{2^{2026}}{2^{22}} = 2^{2004}$, so $k = \boxed{2004}$.

15. A palindrome is a number that reads the same forwards and backwards. Determine the product of the digits of smallest 7-digit palindrome that does not contain the digits 0 or 1 and is divisible by 36.

Solution
Suppose the palindrome is $N = \overline{abcdcba}$, where a, b, c, d are digits at least 2. Since we wish to minimize N, we begin by setting $a = 2$. Since $36|N$, we have $4|N$ and $9|N$, so $4|\overline{b2}$. The smallest possible such b is 3.
Next, we have $9|(2a + 2b + 2c + d)$ or $9|(2c + d + 10)$. Therefore, $2c + d \equiv 8 \pmod 9$. Since we need to minimize c before minimizing d, we set $c = 2$, which forces $d = 4$.
Therefore, the minimum possible N is 2324232. The desired product is $2^6 \cdot 3^2 = \boxed{576}$.

16. Given that a randomly chosen 3-digit integer has at least 2 odd digits, determine the probability that it has exactly 2 odd digits. Express your answer as a common fraction in lowest terms.

Solution
We divide the problem into 2 cases: case 1 is where the 3-digit integer contains 2 odd digits and case 2 is where the 3-digit integer contains 3 odd digits.

Case 1: The 3-digit integer contains 2 odd digits.
If we assume that it is in the hundreds place, then it cannot be 0. Therefore, this yields $4 \cdot 5 \cdot 5 = 100$ 3-digit integers (since there are 5 even and 5 odd digits).
If we assume that it is not in the hundreds place, then there are no restrictions on which even digit it can be, so this gives another $2 \cdot 5 \cdot 5 \cdot 5 = 250$ 3-digit integers.
Altogether, there are $100 + 250 = 350$ 3-digit integers with 2 odd digits.

Case 2: The 3-digit integer contains 3 odd digits.
Since all digits are odd, the hundreds digit must be odd and there are no problems with leading 0s. Therefore, there are $5 \cdot 5 \cdot 5 = 125$ 3-digit integers with 3 odd digits. The desired probability is therefore $\dfrac{350}{350 + 125} = \boxed{\dfrac{14}{19}}$.

17. Charlie has 37 fence panels, each of which are 2 meters long. He wishes to enclose a rectangular area on 3 sides since the wall of a cliff encloses the last side. Given that Charlie does not cut his panels, determine the maximum area that can be enclosed, in square meters.

Solution
Suppose the enclosure has a width of x fences and a length of $37 - 2x$ fences. The total enclosed area in square meters is therefore $4x(37 - 2x) = -8x^2 + 148x$. The maximum of this parabola occurs at $x = \dfrac{148}{2 \cdot 8} = 9.25$. However, since no fences were cut, each side must be an integer number of fences, so we test $x = 9$ and $x = 10$.

If $x = 9$, then the enclosed area is $4 \cdot 9 \cdot (37 - 18) = 684$ square meters and if $x = 10$, the enclosed area is $4 \cdot 10 \cdot (37 - 20) = 680$ square meters. Therefore, the maximum possible area is $\boxed{684}$ square meters.

18. Suppose a and b are real numbers such that $a + b = 4$ and $a^2 + b^2 = 9$. Determine $a^5 + b^5$.

Solution

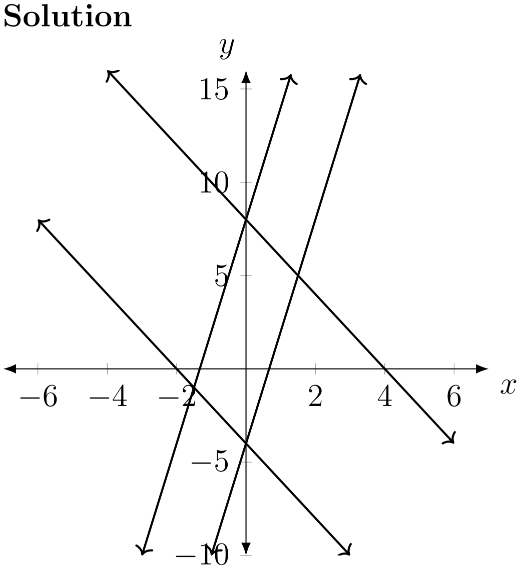
We write $a^5 + b^5 = (a^2 + b^2)(a^3 + b^3) - (a^2b^3 + a^3b^2)$. Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ and $a^2b^3 + a^3b^2 = a^2b^2(a + b)$, the value of ab is quite important.

Note that $ab = \frac{1}{2}((a + b)^2 - (a^2 + b^2)) = \frac{7}{2}$.

We therefore have $a^3 + b^3 = 4\left(9 - \frac{7}{2}\right) = 22$ and $a^2b^3 + a^3b^2 = \left(\frac{7}{2}\right)^2 \cdot 4 = 49$. The value of $a^5 + b^5$ is then $9 \cdot 22 - 49 = \boxed{149}$.

19. The intersections between the lines $y = m_1x + b_1$, $y = m_1x + b_2$, $y = m_2x + b_1$, and $y = m_2x + b_2$ are the vertices of a parallelogram. Given that $b_2 - b_1 = 12$ and $m_2 - m_1 = 8$, determine the area of the parallelogram.

Solution



Note that the y-axis is a diagonal of the parallelogram. Therefore, it divides the parallelogram into 2 congruent triangular halves.

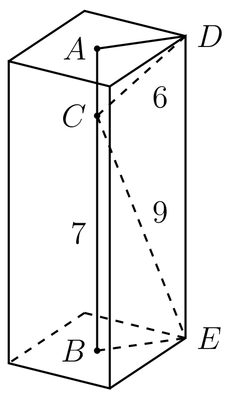
Consider the triangular half of the parallelogram to the right of the y-axis. Its base must be $b_2 - b_1 = 12$. To determine the height, we set $m_2x + b_1 = m_1x + b_2$ and get

$$x = \frac{b_2 - b_1}{m_2 - m_1} = \frac{12}{8} = \frac{3}{2}$$

The area of the parallelogram is therefore $2 \cdot \frac{1}{2} \cdot 12 \cdot \frac{3}{2} = \boxed{18}$.

20. A lightbulb of negligible dimensions is suspended from the center of the ceiling of a tall, square room. Given that the lightbulb is 7 meters, 9 meters, and 6 meters from the center of the floor, a corner at the floor, and a corner at the ceiling, respectively, determine the volume of the room in cubic meters.

Solution



Assume the center of the ceiling is point A, the center of the floor is point B, the lightbulb is at point C, one of the corners at the ceiling is point D (any arbitrary corner), and one of the corners at the floor is E (again any arbitrary corner). We are then given that $BC = 7$, $CE = 9$, and $CD = 6$.

By the Pythagorean theorem on triangle CBE, we have

$$BE^2 = CE^2 - BC^2 = 9^2 - 7^2 = 32$$

Since $BE = AD$, we can write $AC = \sqrt{CD^2 - AD^2} = \sqrt{6^2 - 32} = 2$. Therefore, the height of the room is $AC + BC = 2 + 7 = 9$ and the area of the floor of the room is $2BE^2 = 64$. Therefore, the volume of the room is $2 \cdot 32 \cdot 9 = \boxed{576}$.

21. Evaluate the following expression

$$(\sqrt{13} + \sqrt{14} + \sqrt{15})(\sqrt{13} + \sqrt{14} - \sqrt{15})(\sqrt{13} - \sqrt{14} + \sqrt{15})(-\sqrt{13} + \sqrt{14} + \sqrt{15})$$

Solution

Grouping the first 2 brackets and last 2 brackets, we may write each group as a difference of squares in the following way.

$$((\sqrt{13} + \sqrt{14})^2 - \sqrt{15}^2)(\sqrt{15}^2 - (\sqrt{14} - \sqrt{13})^2)$$

Evaluating each bracket, we have

$$(13 + 14 - 15 + 2\sqrt{182})(15 - 14 - 13 + 2\sqrt{182}) = (2\sqrt{182} + 12)(2\sqrt{182} - 12)$$

Expanding this as a difference of squares, we have $(2\sqrt{182})^2 - 12^2 = 728 - 144 = \boxed{584}$.

22. There are 2 blue marbles, 3 red marbles, and 5 yellow marbles in a bag. Alice draws marbles from this bag at random. Determine the probability that Alice requires at least 8 draws to get all 3 colors of marbles. Express your answer as a common fraction.

Solution

Note that there are $\frac{10!}{5!3!2!} = 2520$ different orders in which Alice can get her 10 marbles, each of which have equal probability. We first determine the number of orders that satisfies the given property.

Since Alice requires at least 8 draws to get all 3 colors, the first 7 marbles can only contain 2 different colors. Therefore, the first 7 marbles are either only blue and yellow or only red and yellow. We consider both cases separately.

Case 1: First 7 marbles are blue and yellow

Since there are only 7 blue and yellow marbles in total, there are $\binom{7}{2} = 21$ different orders to get the first 7 marbles. There are only 3 red marbles left, so there is only 1 order in which Alice can get the last 3 marbles and so there are 21 total orders for this case.

Case 2: First 7 marbles are red and yellow

For this case, the first 7 marbles could either be 3 red and 4 yellow or 2 red and 5 yellow.

If there are 3 red and 4 yellow marbles, then there are $\binom{7}{3} = 35$ different orders to get the first 7 marbles. There are 2 blue and 1 yellow marble left, so there are 3 different orders in which Alice can get the last 3 marbles.

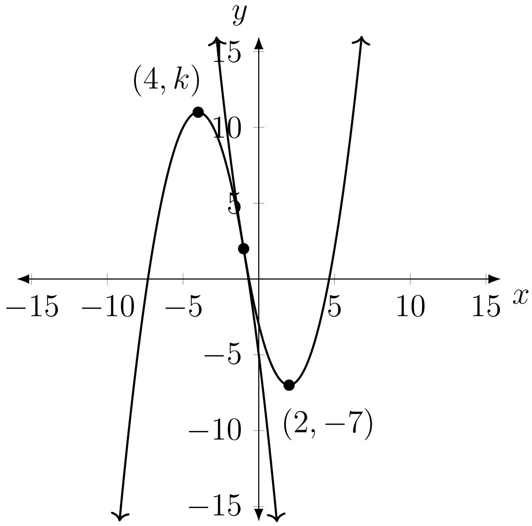
If there are 2 red and 5 yellow marbles, then there are $\binom{7}{2} = 21$ different orders to get the first 7 marbles. There are 2 blue and 1 red marble left, so there are 3 different orders in which Alice can get the last 3 marbles.

In total, this case gives $3(35 + 21) = 168$ different orders.

Putting everything together, there are $21 + 168 = 189$ different possible orders, so the desired probability is $\frac{189}{2520} = \boxed{\frac{3}{40}}$.

23. The quadratic $q_1(x)$ has leading coefficient 1 and its vertex $(2, -7)$, and the quadratic $q_2(x)$ has leading coefficient -1 and its vertex at $(-4, k)$. Given that the graphs of $y = q_1(x)$ and $y = q_2(x)$ are tangent, determine the value of k.

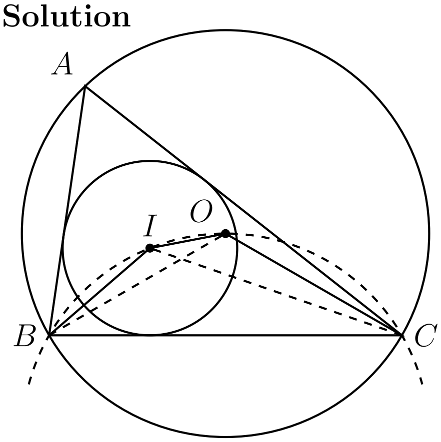
Solution



Since we are given the vertices of q_1 and q_2, their equations in vertex form are
$q_1(x) = (x-2)^2 - 7 = x^2 - 4x - 3, q_2(x) = -(x+4)^2 + k = -x^2 - 8x + (k - 16)$.
Setting $q_1(x) = q_2(x)$, we have $2x^2 + 4x + (13 - k) = 0$. This must only have 1 solution since the graphs of $y = q_1(x)$ and $y = q_2(x)$ are tangent.
Therefore, we have $4^2 - 4 \cdot 2(13 - k) = 0$ or $8k = 104 - 16 = 88$, from which $k = \boxed{11}$.

24. Triangle ABC has circumcenter O and incenter I. Given that quadrilateral $BIOC$ is cyclic, determine the measure of $\angle A$ in degrees.

Solution



Suppose $\angle A = x$. Then by the inscribed angle theorem, $\angle BOC = 2x$.
Note that $\angle B + \angle C = 180^\circ - x$ and $\angle BIC = 180 - \frac{1}{2}(180^\circ - x) = 90^\circ + \frac{1}{2}x$. By the inscribed angle theorem, we have $\angle BOC = \angle BIC$, so $2x = 90^\circ + \frac{x}{2}$ or $\frac{3}{2}x = 90^\circ$. Therefore, $x = \boxed{60^\circ}$.

25. Evaluate the sum $100^3 - 99^3 + 98^3 - 97^3 + \cdots + 2^3 - 1^3$.

Solution

Let the sum be S. We group the terms as $(100^3 - 99^3) + (98^3 - 97^3) + \cdots (2^3 - 1^3)$ and factor each bracket to get

$$S = (100 - 99)(100^2 + 99^2 + 100 \cdot 99) + (98 - 97)(98^2 + 97^2 + 98 \cdot 97) + \cdots + (2 - 1)(2^2 + 1^2 + 2 \cdot 1)$$

We may ignore the first bracket since it always evaluates to 1.

Therefore, $S = \sum_{k=1}^{100} k^2 + (100 \cdot 99 + 98 \cdot 97 + \cdots + 2 \cdot 1)$.

Note that we may express the second bracket as

$$(100^2 - 100) + (98^2 - 98) + \cdots + (2^2 - 2) = (100^2 + 98^2 + \cdots + 2^2) - (100 + 98 + \cdots + 2)$$

Factoring out some constants, this can be further simplified to $4\sum_{k=1}^{50} k^2 - 2\sum_{k=1}^{50} k$.

Therefore,

$$\begin{aligned} S &= \sum_{k=1}^{100} k^2 + 4\sum_{k=1}^{50} k^2 - 2\sum_{k=1}^{50} k \\ &= \frac{100 \cdot 101 \cdot 201}{6} + 4 \cdot \frac{50 \cdot 51 \cdot 101}{6} - 2 \cdot \frac{50 \cdot 51}{2} \\ &= 50 \cdot 101 \cdot 67 + 50 \cdot 101 \cdot 34 - 50 \cdot 51 \\ &= 50(101^2 - 51) \\ &= 50(10150) \\ &= \boxed{507500} \end{aligned}$$

26. For some positive integer $n \leq 2026$, Alice chooses a number a_k from the set $\left\{\frac{k}{k+1}, \frac{k+1}{k}\right\}$ for all positive integers $k \leq n$. Determine the number of possible values of n such that Alice may choose her numbers $a_1, a_2, ...a_n$ such that the product $a_1 a_2 ... a_n = 1$.

Solution

We claim that n must be 1 less than a perfect square. First, we provide a construction for all such n, then we show that no other n are possible.

Construction: For all $1 \leq k \leq \sqrt{n+1} - 1$, we choose $a_k = \frac{k+1}{k}$ and for $\sqrt{n+1} \leq k \leq n$, we choose $a_k = \frac{k}{k+1}$. We have that $\prod_{k=1}^{\sqrt{k+1}-1} a_k = \sqrt{n+1}$ and $\prod_{k=\sqrt{n+1}}^{n} a_k = \frac{1}{\sqrt{n+1}}$, resulting in an overall product of $\frac{\sqrt{n+1}}{\sqrt{n+1}} = 1$. This construction could be motivated by the potentially telescopic nature of these terms.

Proof: If we choose $a_k = \dfrac{k+1}{k}$ for all $1 \leq k \leq n$, then $\displaystyle\prod_{k=1}^{n} a_k = n+1$. Suppose we change the term a_i to $\dfrac{i}{i+1}$ for some integer $1 \leq i \leq n$. The product then becomes $(n+1)\left(\dfrac{i}{i+1}\right)^2$. Clearly, if we change other terms, we always obtain a product of the form $(n+1)\left(\dfrac{p}{q}\right)^2$, where p and q are positive integers. However, since it is possible to choose the changed terms such that $(n+1)\left(\dfrac{p}{q}\right)^2 = 1$, then $\dfrac{p}{q} = \dfrac{1}{\sqrt{n+1}}$, so $\sqrt{n+1}$ must be an integer, as desired.

Since we have $1 \leq n \leq 2026$, we have $2 \leq n+1 \leq 2027$. Since there are 44 perfect squares in this range, there are $\boxed{44}$ possible values of n.

Bull's Eye

Problem Solving

1. Given that 2 cows and 3 goats have the same value as 1 cow and 5 goats, then 2026 goats is equivalent to k cows. Determine the value of k.

Solution

Suppose the value of 1 cow is c and the value of 1 goat is g. We have $2c + 3g = c + 5g$ or $c = 2g$. Therefore, $2026g = 1013c$ so 2026 goats is equivalent to $\boxed{1013}$ cows.

2. A band teacher realizes that he can arrange his class into x rows with y students in each row as well as $x - 3$ rows with $y + 4$ students in each row. Given there are between 50 and 100 students in the marching band, determine number of students in the marching band.

Solution

We have $xy = (x - 3)(y + 4) = xy + 4x - 3y - 12$. Cancelling xy from both sides, we have $4x = 3y + 12$ or $x = \frac{3}{4}y + 3$.

Therefore, we have $4|y$. Note that if $y = 4$, then $x = 6$, so $xy = 24 < 50$. If $y = 8$, then $x = 9$, so $xy = 72$ which is in the correct range. If $y = 12$, then $x = 12$, so $xy = 144 > 100$. Therefore, there are $\boxed{72}$ students in the marching band.

3. Suppose $\{a_n\}$ and $\{b_n\}$ are arithmetic sequences with common differences d_a and d_b, respectively such that $a_{20} = b_{26}$ and $a_{202} = b_6$. Determine the ratio $\frac{d_b}{d_a}$. Express your answer as a common fraction in lowest terms.

Solution

We have that the general terms of the 2 sequences a_n and b_n are $a_0 + (n-1)d_a$ and $b_0 + (n-1)d_b$, respectively.

Therefore, we have the following equations.

$$a_0 + 19d_a = b_0 + 25d_b$$
$$a_0 + 201d_a = b_0 + 5d_b$$

Subtracting the 2 equations gives $182d_a = -20d_b$, so $\frac{d_b}{d_a} = \boxed{-\frac{91}{10}}$.

4. A rectangular prism has sides of integer lengths and a volume of 126. A new rectangular prism is created by extending all the sides of the previous rectangular prism by 1. Determine the minimum possible volume of the new prism.

Solution

Assume the side lengths of the rectangular prism are a, b, and c. The total increase in volume when extending the sides is

$$(a+1)(b+1)(c+1) - abc = (ab+bc+ac) + (a+b+c) + 1$$

We wish to minimize this increase.

By AM-GM, we know that both $ab+ac+bc$ and $a+b+c$ are minimized when $a = b = c$. However, since $\sqrt[3]{126} \approx 5$ is not an integer, this is not possible. Instead, we have to find 3 positive integers that multiply to 126 that are as close as possible.

Since $126 = 2 \cdot 3^2 \cdot 7$, we know one of the dimensions must be some multiple of 7 to account for that factor. Since $7 > \sqrt[3]{126}$, we set 7 itself to be one of the dimensions.
The other 2 dimensions must multiply to 18. The closest we can get to equal dimensions using integers is 3 by 6.

Therefore, a prism that is 3 by 6 by 7 will minimize the increase in volume. The prism has dimensions 4 by 7 by 8 and so has volume $4 \cdot 7 \cdot 8 = \boxed{224}$.

5. In a group of 10 students, a teacher selects 1 to be president, 2 to be vice presidents, and 3 to be secretaries. Given that no student has more than 1 role, determine the number of ways this can this be done.

Solution

Without loss of generality, assume we choose the president first, then the vice presidents, then the secretaries. There are $\binom{10}{1} = 10$ ways to choose the president among the 10 students, $\binom{9}{2} = 36$ ways to choose the vice presidents among the remaining 9 students, and $\binom{7}{3} = 35$ ways to choose the secretaries among the remaining 7 students.
Therefore, there are $10 \cdot 36 \cdot 35 = \boxed{12600}$ ways to assign the roles to the students.

6. Alice has 6 square tiles with identical dimensions, 2 black and 4 white. 2 arrangements are considered the same if one can be rotated to obtain the other. Determine the number of different arrangements that Alice can make.

Solution

Note the rectangle must have dimensions 1 by 6, 2 by 3, 3 by 2, or 6 by 1. However, each 6 by 1 rotationally distinct rectangle can be generated to rotating a 1 by 6 rectangle by 90 degrees, and similarly for the 3 by 2 rotationally distinct rectangles. Therefore, we focus on the 1 by 6 and 2 by 3 cases and we need only consider rotations by 180 degrees.

Case 1: 1 by 6 rectangle

Note that rotating 180 degrees in this case amounts to reflecting about the line between the 3rd and 4th tiles. Therefore, each 1 by 6 rectangle that is symmetric about it has no other rotationally equivalent rectangle. There are 3 such rectangles.
All other rectangles have exactly 1 rotationally equivalent rectangle. Since there are $\binom{6}{2} - 3 = 12$ other rectangles, 6 of them are rotationally distinct, for a total of 9.

Case 2: 2 by 3 rectangle

Note that when the rectangle is rotated, the top left and bottom right tiles switch, the bottom left and top right tiles switch, and the 2 middle tiles switch. Therefore, there are again 3 rectangles that are invariant under the rotation and 6 other rotationally distinct rectangles, for a total of 9.
Putting everything together gives $\boxed{18}$ rectangles.

7. Determine the number of positive integers n such that $n^3 + 7n^2 - 5n + 13$ is a multiple of $n - 5$.

Solution

We write $n^3 + 7n^2 - 5n + 13 = (n-5)(n^2 + an + b) + c$ and equate coefficients. Expanding out the right hand side, we have $n^3 + (a-5)n^2 + (b-5a)n + (c-5b)$. Therefore, we obtain the system

$$a - 5 = 7 \quad (4)$$
$$b - 5a = -5 \quad (5)$$
$$c - 5b = 13 \quad (6)$$

Rearranging equation (4), we have $a = 7 + 5 = 12$. Substituting into (5) gives $b = 5 \cdot 12 - 5 = 55$. Substituting into (6) gives $c = 5 \cdot 55 + 13 = 288$.
Therefore, $n^3 + 7n^2 - 5n + 13 = (n-5)(n^2 + 12n + 55) + 288$. Since the left hand side is a multiple of $n - 5$ and $(n-5)(n^2 + 12n + 55)$ is always a multiple of $n - 5$, we have that 288 must be a multiple of $n - 5$ or $n - 5$ must be a factor of 288.

Since $n \geq 1$, $n - 5 \geq -4$. Since $288 = 2^5 \cdot 3^2$, it has $(5+1)(2+1) = 18$ factors, so $n - 5$ can take on 18 different positive values as well as $-1, -2, -3$, and -4, for a total of $\boxed{22}$ different positive integers n.

8. Assume n is a positive integer with 18 distinct positive factors, while n^2 has 75 distinct positive factors. Determine the number of distinct positive factors of n^3.

Solution

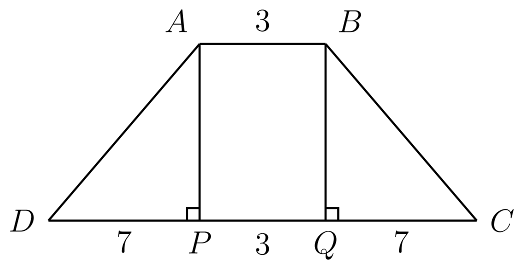
Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2 \ldots p_k$ be the prime factors of n and $a_1, a_2 \ldots a_k > 0$ are the corresponding exponents. n must have $(a_1+1)(a_2+1)\cdots(a_k+1) = 18$ factors and n^2 must have $(2a_1+1)(2a_2+1)\cdots(2a_k+1) = 75$ factors. Dividing one by the other gives

$$\frac{(2a_1+1)(2a_2+1)\cdots(2a_k+1)}{(a_1+1)(a_2+1)\cdots(a_k+1)} = \frac{75}{18} < \frac{(2a_1+2)(2a_2+2)\cdots(2a_k+2)}{(a_1+1)(a_2+1)\cdots(a_k+1)} = 2^k$$

Note that $\frac{75}{18} > 4 = 2^2$, so $k \geq 3$. However, $18 = 2 \cdot 3^2$ (product of 3 prime factors), so we may express 18 as the product of at most 3 integers greater than 1. Since all of $a_1+1, a_2+1, ..., a_k+1 > 1$, we have $k \leq 3$ and so $k = 3$. Therefore, the prime factorization of n must be of the form $p_1^2 p_2^2 p_3$. Therefore, n^3 has $(3(2)+1)(3(2)+1)(3(1)+1) = \boxed{196}$ factors.

9. An isosceles trapezoid has $AB \parallel CD$, $AB = 3$, $CD = 17$, and an area of 240. Determine the perimeter of $ABCD$.

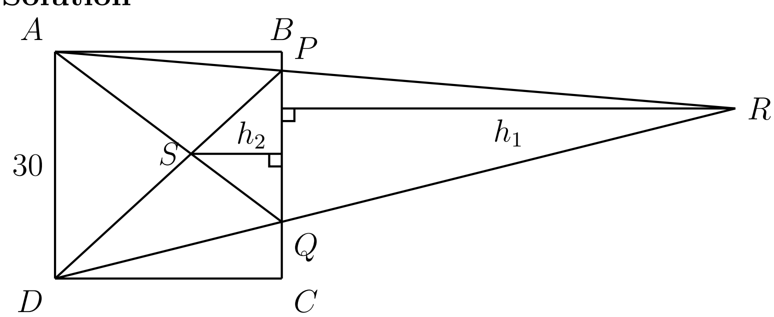
Solution



The height of trapezoid $ABCD$ must be $\frac{2 \cdot 240}{3+17} = 24$.
Suppose points P and Q are on CD such that $AP \perp CD$ and $BQ \perp CD$. Since $AD = BC$ and $AP = BQ$ and both triangles APD and BQC are right triangles, we must have that triangle APD is congruent to BQC, so $DP = CQ$.
However, since $ABQP$ is a rectangle, $AB = PQ = 3$, so $2DP + 3 = CD = 17$ or $DP = CQ = 7$.
Therefore, $AD = BC = \sqrt{AP^2 + DP^2} = \sqrt{7^2 + 24^2} = 25$, so the perimeter is $3+17+25+25 = \boxed{70}$.

10. Square $ABCD$ has side length 30. There exists points P and Q on BC such that $PQ = 20$. Let R be the intersection of AP and DQ and let S be the intersection of AQ and DP. Determine the area of quadrilateral $PRQS$.

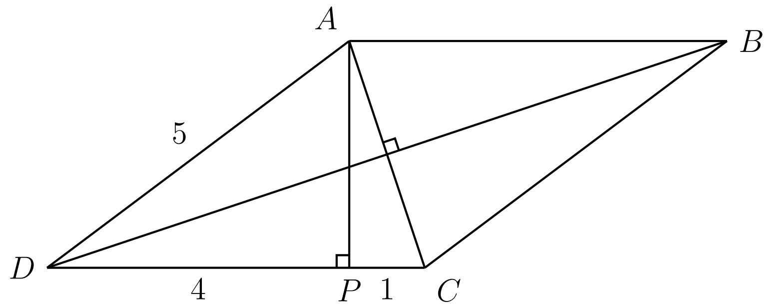
Solution



We can divide quadrilateral $PRQS$ into triangles PQR and PQS. Suppose the perpendicular distance from R to PQ is h_1. Since $PQ \parallel AD$, we have $\angle RPQ = \angle RAD$ and $\angle RQP = \angle RDA$. Therefore, triangle PQR is similar to ADR so we have $\frac{h_1}{h_1 + 30} = \frac{2}{3}$, so $h_1 = 60$. The area of PQR is $\frac{1}{2} \cdot 20 \cdot 60 = 600$. Suppose the perpendicular distance from S to PQ is h_2. Since $PQ \parallel AD$, we have $\angle SPQ = \angle SDA$ and $\angle SQP = \angle SAD$. Therefore, triangle PQS is similar to triangle DAS, so we have $\frac{h_2}{30 - h_2} = \frac{2}{3}$, so $h_2 = 12$. The area of PQS is $\frac{1}{2} \cdot 20 \cdot 12 = 120$. Putting everything together, the area of $PRQS$ is $600 + 120 = \boxed{720}$.

11. In rhombus $ABCD$, suppose that point P is on CD such that $AP \perp CD$. Given that $AB = 5PC$, then determine the value of $\frac{BD}{AC}$.

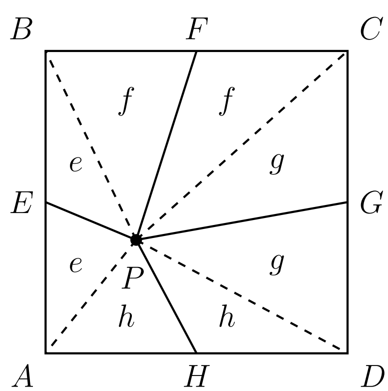
Solution



Suppose $PC = 1$. Then we have $AB = BC = CD = AD = 5$ and $PD = 4$.
By the Pythagorean theorem, $AP = \sqrt{AD^2 - PD^2} = \sqrt{5^2 - 4^2} = 3$, so the area of $ABCD$ is $3 \cdot 5 = 15$.
However, we must also have that $\frac{1}{2} AC \cdot BD = 15$ or $AC \cdot BD = 30$. By the Pythagorean theorem, $AC = \sqrt{AP^2 + CP^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$, so
$BD = \frac{30}{\sqrt{10}} = 3\sqrt{10}$. Therefore, $\frac{BD}{AC} = \boxed{3}$.

12. In square $ABCD$, assume points E, F, G, H are the midpoints of AB, BC, CD, AD, respectively. Let P be a point in the interior of $ABCD$. Given that the areas of quadrilaterals $AEPH$ and $CGPF$ are 31 and 67, respectively, determine the length of AB.

Solution



Draw AP, BP, CP, DP and let $[\ldots]$ denote area.
Since $AE = EB$, we have $[APE] = [BPE] = e$. Similarly, $[BPF] = [CPF] = f$, $[CGP] = [DGP] = g$, and $[DHP] = [AHP] = h$.
Note that $[AEPH] = e + h$ and $[CGPF] = f + g$ and $[ABCD] = 2(e + f + g + h)$. Therefore, $AB^2 = [ABCD] = 2([AEPH] + [CGPF]) = 2(31 + 67) = 196$, so
$AB = \sqrt{196} = \boxed{14}$.

Co-op

1. Evaluate $2+\dfrac{1}{0+\frac{1}{2+\frac{1}{6}}}$. Express your answer as a common fraction in lowest terms.

Solution

Evaluating, we have $2+\dfrac{1}{0+\frac{1}{2+\frac{1}{6}}}=2+\dfrac{1}{0+\frac{6}{13}}=2+\dfrac{13}{6}=\boxed{\dfrac{25}{6}}$.

2. If x and y are real numbers such that $\dfrac{3x+4}{5y+6}=\dfrac{2}{3}$, determine the value of $\dfrac{y}{x}$. Express your answer as a common fraction in lowest terms.

Solution

Cross-multiplying, we have $3(3x+4)=2(5y+6)$ or $9x+12=10y+12$. Therefore, $9x=10y$, so $\dfrac{y}{x}=\boxed{\dfrac{9}{10}}$.

3. The current ages of Mary and Jane are in the ratio 5:7. In 4 years, their ages will be in the ratio 3:4. Determine Jane's current age.

Solution

Assume Mary's current age is $5x$ and Jane's current age is $7x$. In 4 years, their ages will be $5x+4$ and $7x+4$, respectively, so we have $\dfrac{5x+4}{7x+4}=\dfrac{3}{4}$. Cross multiplying, we have $4(5x+4)=3(7x+4)$ or $x=4$. Therefore, Jane's current age is $7\cdot 4=\boxed{28}$.

4. Alice and Bob are each running 100 times around the same track at constant speeds. To complete each lap, Alice takes 2 minutes, while Bob takes 2 minutes and 15 seconds. Over the course of the 100 laps, determine the number of times that Alice laps Bob.

Solution

Suppose that after Bob runs $x-1$ laps, Alice runs x laps. Since 2 minutes is 120 seconds and 2 minutes 15 seconds is 135 seconds, we have $\dfrac{x}{x-1}=\dfrac{135}{120}$, from which $x=9$. Therefore, Alice will lap Bob every 9 laps, so over the course of the 100 laps, she will lap Bob $\left\lfloor\dfrac{100}{9}\right\rfloor=\boxed{11}$ times.

5. A bag contains 5 red and 7 yellow marbles. Alice draws 3 marbles from the bag without replacement. Determine the probability that the third marble is red. Express your answer as a fraction in lowest terms.

Solution

Let Y denote a yellow marble and R denote a red marble. We consider the cases YYR, YRR, RYR, and RRR.

The first case occurs with probability $\dfrac{7 \cdot 6 \cdot 5}{12 \cdot 11 \cdot 10} = \dfrac{210}{12 \cdot 11 \cdot 10}$.

The second case occurs with probability $\dfrac{7 \cdot 5 \cdot 4}{12 \cdot 11 \cdot 10} = \dfrac{140}{12 \cdot 11 \cdot 10}$.

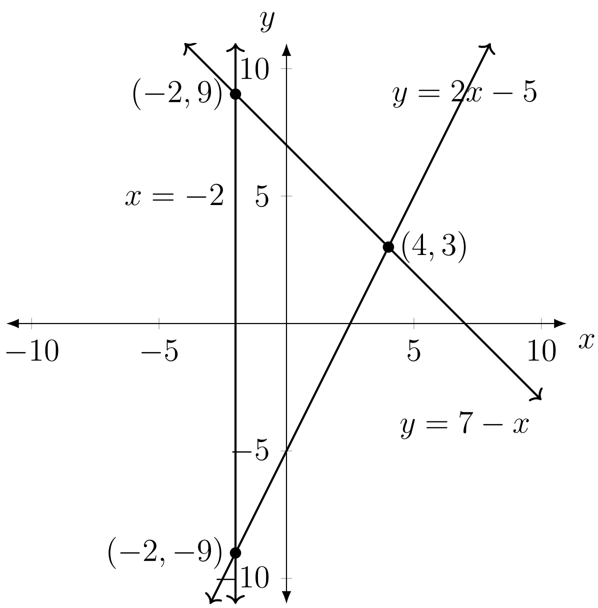
The third case occurs with probability $\dfrac{5 \cdot 7 \cdot 4}{12 \cdot 11 \cdot 10} = \dfrac{140}{12 \cdot 11 \cdot 10}$.

The fourth case occurs with probability $\dfrac{5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10} = \dfrac{60}{12 \cdot 11 \cdot 10}$.

Therefore, the total probability is $\dfrac{210 + 140 + 140 + 60}{12 \cdot 11 \cdot 10} = \dfrac{550}{12 \cdot 11 \cdot 10} = \boxed{\dfrac{5}{12}}$.

6. Determine the area enclosed by the graph of $(2x - y - 5)(x + y - 7)(x + 2) = 0$.

Solution



Note that the equation is satisfied precisely when $y = 2x - 5$, $y = 7 - x$, or $x = -2$. The 3 lines bound a triangle and we determine the coordinates of the vertices.

Let A be the intersection of $y = 2x - 5$ with $x = -2$ and let B be the intersection of $y = 7 - x$ and $x = -2$. For these 2 points, we simply substitute $x = -2$ to get the corresponding y-coordinates. Therefore, $A = (-2, 2(-2) - 5) = (-2, -9)$ and $B = (-2, 7 - (-2)) = (-2, 9)$.

Let C be the intersection of $y = 2x - 5$ and $y = 7 - x$. Setting them equal we have $2x - 5 = 7 - x$ or $3x = 12$ or $x = 4$. Note that the perpendicular distance from C to AB is $4 + 2 = 6$ and the length of AB is $9 + 9 = 18$, so the area of ABC is $\dfrac{1}{2} \cdot 6 \cdot 18 = \boxed{54}$.

7. Suppose $\{a_n\}$ is a sequence satisfying $a_1 = 10$ and $a_{n+1} = \dfrac{a_n + 1}{1 - a_n}$. Determine a_{2026}. Express your answer as a common fraction in lowest terms.

Solution

Writing out the first few terms, we have

$$\begin{aligned} a_2 &= \frac{10+1}{1-10} = -\frac{11}{9} \\ a_3 &= \frac{1-\frac{11}{9}}{1+\frac{11}{9}} = -\frac{1}{10} \\ a_4 &= \frac{1-\frac{1}{10}}{1+\frac{1}{10}} = \frac{9}{11} \\ a_5 &= \frac{\frac{9}{11}+1}{1-\frac{9}{11}} = 10 \end{aligned}$$

Since each term depends only on 1 term before it and $a_1 = a_5$, the sequence repeats with a period of 4. Therefore, $a_{2026} = a_2 = \boxed{-\frac{11}{9}}$.

8. Alice and Bob were walking to their boarding gates at an airport with constant speeds along parallel, straight paths. At $t = 0$ seconds, Alice passes Bob. At $t = 40$ seconds, Bob gets on a long conveyor belt that travels at 2m/s while continuing to walk at his constant speed relative to the conveyor. At $t = 90$ seconds, Bob passes Alice. Determine the difference between Alice and Bob's speeds in km/h.

Solution

Suppose Alice's and Bob's constant speeds are v_a and v_b in m/s, respectively and let $v_a - v_b = v_d$. At $t = 40$ seconds, Alice is ahead of Bob by $40v_d$ meters and by $t = 90$ seconds, Bob must catch up by $50(2 - v_d)$ meters.

Since these 2 distances are equal, we have $40v_d = 50(2 - v_d)$ or $9v_d = 10$m/s. Therefore, $v_d = \frac{10}{9}$m/s. This is equal to $\frac{10}{9} \cdot \frac{18}{5} = \boxed{4\text{km/h}}$.

9. Given that $\frac{17}{20} < \frac{p}{q} < \frac{6}{7}$, where p and q are positive integers, determine the minimum possible value of $p + q$.

Solution

Multiplying out both inequalities gives $17q < 20p$ and $7p < 6q$ or $17q + 1 \leq 20p$ and $7p + 1 \leq 6q$. Multiplying the first inequality by 7 and the second by 20 and adding, we have $7(17q+1)+20(7p+1) = 119q + 140p + 27 \leq 140p + 120q$. Therefore, we have $q \geq 27$.

If we set $q = 27$, then we note that the corresponding p is 23, so the minimum possible $p + q$ is $23 + 27 = \boxed{50}$.

10. The equation $3^{4052x+2} + 81 = 3^{2026x+5}$ has real solutions x_1 and x_2. Determine the value of $\dfrac{1}{x_1 + x_2}$.

Solution

Let $y = 3^{2026x}$. Then the given equation becomes $9y^2 - 243y + 81 = 0$ or $y^2 - 27y + 9 = 0$. Assume y_1 and y_2 are the roots of this quadratic. Then $y_1 y_2 = 3^{2026x_1} \cdot 3^{2026x_2} = 3^{2026(x_1+x_2)}$. However, we also know by Vieta's formulas that $y_1 y_2 = 9$, so $2026(x_1 + x_2) = 2$, or $\dfrac{1}{x_1 + x_2} = \boxed{1013}$.

11. Determine the remainder when $9^{2026} + 11^{2026}$ is divided by 1000.

Solution

We may write $9^{2026} + 11^{2026}$ as $(10-1)^{2026} + (10+1)^{2026}$. Using the binomial expansion, we have $(10-1)^{2026} + (10+1)^{2026} = \sum_{k=0}^{2026} (-1)^{2026-k} \binom{2026}{k} 10^k + \binom{2026}{k} 10^k$.

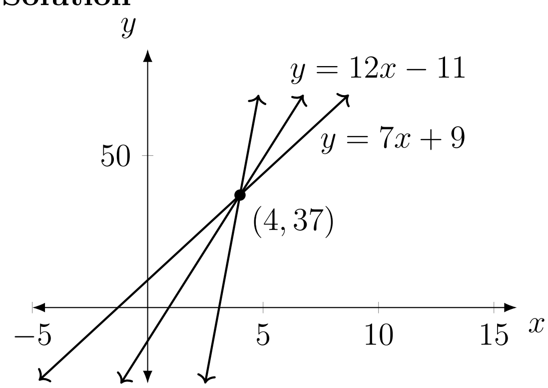
Note that for odd k, $(-1)^{2026-k} = -1$, so the term goes to 0 and for even k. Therefore, the sum can be simplified to $2 \sum_{k=0}^{1013} \binom{2026}{2k} 10^{2k}$.

Since we are going to take the sum mod 1000, we may disregard the terms corresponding to all $k \geq 2$ since they all have at least a factor of 10^4. Therefore, $2 \sum_{k=0}^{1013} \binom{2026}{2k} 10^{2k} \equiv 2 + 200 \binom{2025}{2} \pmod{1000}$.

However, note that $200 \binom{2025}{2} = 200 \cdot \dfrac{2025 \cdot 2024}{2} = 1000 \cdot 405 \cdot 1012 \equiv 0 \pmod{1000}$. Therefore, the remainder is simply $\boxed{2}$.

12. Suppose the line $y = mx - b$ is the image of the line $y = 7x + 9$ after reflecting about the line $y = 12x - 11$. Determine the value of $m + b$.

Solution

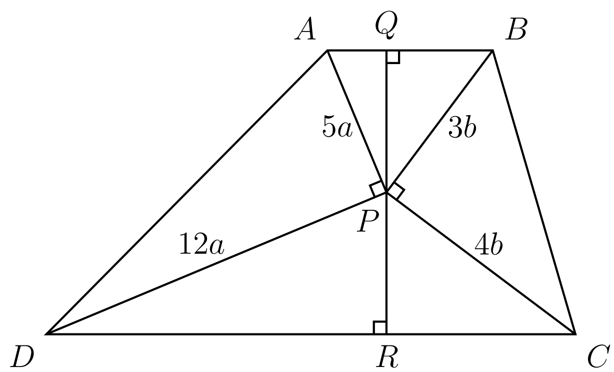


First, we note that $y = mx - b$ should pass through the intersection of $y = 7x + 9$ and $y = 12x - 11$. Setting them equal, we have $7x + 9 = 12x - 11$, from which $x = 4$ and $y = 7(4) + 9 = 37$. Now we shift the intersection to the origin to determine m and shift back to $(4, 37)$ to determine b.

This question is now equivalent to determining the value of m such that $y = 12x$ is an angle bisector of the lines $y = 7x$ and $y = mx$. In particular, if we let $A = (0, 0)$, $B = (1, 7)$, $C = (1, m)$, and $D = (1, 12)$, then AD bisects $\angle BAC$. Clearly, $BD = 5$ and $CD = m - 12$. By the Pythagorean theorem, we also have $AB = \sqrt{7^2 + 1^2} = \sqrt{50}$ and $AC = \sqrt{m^2 + 1}$. By the angle bisector theorem, we have $\sqrt{\dfrac{m^2+1}{50}} = \dfrac{m-12}{5}$. Squaring both sides gives $\dfrac{m^2+1}{50} = \dfrac{(m-12)^2}{25}$ or $m^2 + 1 = 2(m - 12)^2$. Expanding out and collective like terms gives $m^2 - 48m + 287 = 0$
or $(m - 7)(m - 41) = 0$. Clearly, the root we want is $m = 41$. Now we arrange for $y = 41x - b$ to pass through $(4, 37)$. We have $37 = 41 \cdot 4 - b$, so $b = 4 \cdot 41 - 37 = 127$. Therefore, $m + b = 41 + 127 = \boxed{168}$.

13. In trapezoid $ABCD$, $AB || CD$ and the angle bisectors of $\angle A$, $\angle B$, $\angle C$, and $\angle D$ all meet at P. Given that $\dfrac{AP}{DP} = \dfrac{5}{12}$, $\dfrac{BP}{CP} = \dfrac{3}{4}$, and the area of trapezoid $ABCD$ is 90, determine the perimeter of trapezoid $ABCD$.

Solution



Since $\angle PAD = \dfrac{1}{2}\angle A$ and $\angle PDA = \dfrac{1}{2}\angle D$ and $\angle A + \angle D = 180°$, we have
$\angle PAD + \angle PDA = \frac{1}{2} \cdot 180° = 90°$. Therefore, $\angle APD$ is a right angle. Moreover, since $\dfrac{AP}{DP} = \dfrac{5}{12}$, we know that triangle APD is similar to a 5-12-13 right triangle. We may similarly show that $\angle BPC$ is a right angle and triangle BPC is similar to a 3-4-5 right triangle.

Assume $AP = 5a$, $DP = 12a$, and $AD = 13a$. Similarly, assume $BP = 3b$, $CP = 4b$, and $BC = 5b$. Further assume that Q is a point on AB such that $PQ \perp AB$ and R is a point on CD such that $PR \perp CD$.
Since $\angle PAD = \angle PAQ$ and both triangles PAD and PAQ are right triangles, triangles QAP and PAD are similar, so $\dfrac{PQ}{AP} = \dfrac{PD}{AD} = \dfrac{12}{13}$, so $PQ = \dfrac{12}{13} \cdot 5a = \dfrac{60}{13}a$ and $AQ = \dfrac{5}{12} \cdot 5a = \dfrac{25}{12}a$.
Similarly, we may show that triangles RPD and PAD are similar, so
$PR = \dfrac{5}{13} \cdot 12a = \dfrac{60}{13}a$ and $DR = \dfrac{12}{13} \cdot 12a = \dfrac{144}{13}a$.

Repeating on the other half of the trapezoid, we have that triangles QBP and PBC are similar, so $PQ = \frac{4}{5} \cdot 3b = \frac{12}{5}b$ and $QB = \frac{3}{5} \cdot 3b = \frac{9}{5}b$. Triangles RPC and PBC are similar, so $PR = \frac{3}{5} \cdot 4b = \frac{12}{5}b$ and $CR = \frac{4}{5} \cdot 4b = \frac{16}{5}b$.

Setting the 2 expressions for PQ equal, we have $\frac{60}{13}a = \frac{12}{5}b$ or $a = \frac{13}{25}b$. The height of the trapezoid is $\frac{24}{5}b$ and the sum of the bases $AB + CD$ is

$13a + 5b = \frac{169 + 125}{25}b = \frac{294}{25}b$.

Therefore, we have $\frac{1}{2} \cdot \frac{24}{5}b \cdot \frac{294}{25}b = \frac{2^3 \cdot 3^2 \cdot 7^2}{5^3}b^2 = 90$.

Cancelling some factors gives $\frac{14^2}{5^2}b^2 = 25$, so $b = \frac{25}{14}$.

Note that the perimeter of $ABCD$ is simply $2(AB + CD) = 2 \cdot \frac{294}{25}b$. Therefore, the desired perimeter is $2 \cdot \frac{294}{25} \cdot \frac{25}{14} = \boxed{42}$.

14. Let S be the infinite sum $\frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \cdots$. Determine the remainder when $\lfloor 10^{2026}S \rfloor$ is divided by 1000, where $\lfloor x \rfloor$ is the greatest integer that is at most x.

Solution

Note that a string of n 9s is equivalent to $10^n - 1$. Therefore, we may express the sum as $\sum_{k=1}^{\infty} \frac{1}{10^k - 1}$.

Dividing through by 10^k on each term in the summation, we get the sum $\sum_{k=1}^{\infty} \frac{\frac{1}{10^k}}{1 - \frac{1}{10^k}}$, which is suggestive of the formula for the sum of an infinite geometric series. Indeed, the kth term in the original summation can be expressed as the infinite geometric series $\sum_{i=1}^{\infty} \frac{1}{10^{ki}}$. Therefore, we may write $S = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{10^{ki}}$.

Note that the ith geometric series in the overall sum only contains a term of $\frac{1}{10^m}$ if m is a multiple of i, or equivalently, if i is a factor of m. Therefore, the number of terms of $\frac{1}{10^m}$ is exactly the number of factors of m for all positive integers m, so $S = \sum_{m=1}^{\infty} \frac{d(m)}{10^m}$, where $d(m)$ is the number of factors of m.

Now $10^{2026}S = \sum_{m=1}^{\infty} d(m)10^{2026-m}$. Taking the floor removes all terms with $m \geq 2027$, and taking mod 1000 removes all terms with $m \leq 2023$. Therefore, the remainder is $100d(2024) + 10d(2025) + d(2026) = 1600 + 150 + 4 \equiv \boxed{754} \pmod{1000}$.

15. Given that positive real x and y satisfies the system

$$\begin{aligned}(\sqrt{x}+4)(\sqrt{y}+4) &= 97\\(\sqrt{x}+\sqrt{y}-3)(\sqrt{xy}-3) &= 260\end{aligned}$$

determine the value of $x+y$.

Solution

Label the given equations as

$$(\sqrt{x}+4)(\sqrt{y}+4) = 97 \tag{7}$$
$$(\sqrt{x}+\sqrt{y}-3)(\sqrt{xy}-3) = 260 \tag{8}$$

Expanding out (7) gives $\sqrt{xy}+4(\sqrt{x}+\sqrt{y})+16=97$ or $\sqrt{xy}+4(\sqrt{x}+\sqrt{y})=81$. Looking at (8), it would make sense to make the substitution $a=\sqrt{x}+\sqrt{y}$ and $b=\sqrt{xy}$. Therefore, we may rewrite the system as

$$4a+b=81 \tag{9}$$
$$(a-3)(b-3)=260 \tag{10}$$

Rearranging (9) gives $b=81-4a$ and substituting into (7) gives $(a-3)(78-4a)=260$. Expanding it out, we have

$$-4a^2+90a-234=260$$
$$2a^2-45a+247=0$$
$$(2a-19)(a-13)=0 \Rightarrow a=\frac{19}{2},13$$

If $a=\frac{19}{2}$, then $b=81-4\cdot\frac{19}{2}=43$. Recalling how we defined a and b to be in terms of x and y, we have the system

$$\sqrt{x}+\sqrt{y}=\frac{19}{2}$$
$$\sqrt{xy}=43$$

Recalling Vieta's formulas, we have that $\sqrt{x}$ and $\sqrt{y}$ are the roots of the quadratic $m^2-\frac{19}{2}m+43$. However, the discriminant is $\left(\frac{19}{2}\right)^2-4(43)=90.25-172<0$. Therefore, this quadratic yields no real solutions (x,y).

If $a=13$, then $b=81-4(13)=29$. Just as in the previous case, we the system

$$\sqrt{x}+\sqrt{y}=13$$
$$\sqrt{xy}=29$$

and so $\sqrt{x}$ and $\sqrt{y}$ are the roots of the quadratic $m^2-13m+29$. The discriminant of this quadratic is $13^2-4(29)=53>0$. Therefore, this quadratic yields real solutions (x,y).
Finally, we have $(\sqrt{x}+\sqrt{y})^2-2\sqrt{xy}=x+y=13^2-2\cdot 29=\boxed{111}$.